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ON THE ZERO SET OF A HOLOMORPHIC ONE-FORM ON A COMPACT COMPLEX MANIFOLD

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ABSTRACT. On any compact complex surface M, divisors of nonnegative self-intersection which are contained in the zero set (or in the integral set) of a holomorphic 1-form are shown to induce a fibration of M onto a Riemann surface. This result is extended to higher dimensions for M projective. Applications to zero sets of holomorphic 1-forms on surfaces are given.

Introduction. The zero set of a vector field on a compact manifold M has for some time been known to convey structural information about M. In the compact holomorphic category, zero sets of holomorphic vector fields provide extensions of the Gauss-Bonnet theorem, and, for M Kähler, the dimension of the zero set of a holomorphic vector field relates to cohomological vanishing (via the Carrell-Lieberman theorem [**2**]).

In the dual case, the zero set (and more generally the integral set) of a holomorphic 1-form can convey structural information for a compact complex manifold M. In Theorem 1, we show that structural information is obtained for any compact surface which carries a holomorphic 1-form pulling back to zero on a divisor D of nonnegative self-intersection. In Theorem 2, we extend this result to the case M is higher dimensional and projective.

We state

THEOREM 1. *Let S be any compact complex surface having a holomorphic 1-form ϕ. If D is any divisor on S satisfying $D \cdot D \geq 0$ and ϕ pulls back to zero on D, then there exists a holomorphic map $f\colon S \to R$ onto a compact Riemann surface satisfying*

(1) *$\phi = f^*(\phi_R)$ for some $\phi_R \in H^0(R, \Omega^1_R)$.*

(2) *f has connected fibers.*

(3) *Each component of D is setwise contained in a fiber of f.*

(4) *Each connected component of D is a rational multiple of the natural divisor associated to the fiber of f containing it, and so $D \cdot D = 0$.*

Notationally, $D \cdot D$ is the intersection pairing of D with itself. Ω^1_M denotes the sheaf of germs of holomorphic 1-forms on the manifold M. A holomorphic 1-form ϕ is said to pull back to zero on a divisor $D = \sum_{j=1}^k n_j D_j$ if and only if $i_j^* \phi = 0$ on D_j for each j, where $i_j\colon D_j \to S$ is inclusion. We remark that if ϕ vanishes on D then it necessarily pulls back to zero on D, but the converse need not hold.

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Theorem 1 and its converse characterize the existence of a holomorphic map $f\colon S \to R$ from a compact complex surface onto a compact Riemann surface of genus $g(R) \geq 1$. The converse of Theorem 1 is easily seen to hold: if $f\colon S \to R$ is holomorphic onto a compact Riemann surface of genus $g(R) \geq 1$, and if D is the divisor associated to any fiber of f, then $D \cdot D = 0$ and $f^*(\phi_R)$ pulls back to zero on D for any $\phi_R \in H^0(R, \Omega^1_R)$. Note, therefore, that the condition that $D \cdot D \geq 0$ in the theorem is a natural one.

§§1 and 2 give the proof of Theorem 1 in the Kähler and non-Kähler cases. §3 extends Theorem 1 to the higher dimensional, projective case. §4 gives some remarks and two applications in the surface setting. We mention these here. If a curve C in a compact surface S is contained in the zero set of a holomorphic 1-form then its genus $g(C)$ is bounded above in terms of Chern numbers of S. As a partial converse, if C is of sufficiently low genus and has nonnegative self-intersection then C has zero self-intersection and must be contained in the zero set of a holomorphic 1-form on S.

1. Proof of Theorem 1: The Kähler case. In this section we utilize an idea of A. J. Sommese to use the Albanese mapping and the characterization of an exceptional set in S in terms of the negative definiteness of its self-intersection matrix, in showing the Kähler case. We remark that the algebraic case has been shown by C. P. Ramanujam [**6**] and by F. Catanese [**3**, p. 510].

Assume that S is Kähler; for the moment it is assumed that $D = \sum_{j=1}^k n_j D_j$ is connected. Without any loss of generality, we also assume S is free of exceptional curves of the first kind.

Let $\alpha_S\colon S \to A(S)$ be the Albanese mapping of S into its Albanese torus $A(S)$. The method is to produce a subtorus T_1 of $A(S)$ with $\alpha_S(\bigcup_{j=1}^k D_j) \subset T_1$, but with $\alpha_S(S) \not\subset T_1$. Defining $T := A(S)/T_1$ as the quotient torus and $Q\colon A(S) \to T$ as the quotient map, it will be shown that $Q \circ \alpha_S(S)$ in T is one-dimensional. Taking the Stein factorization of $Q \circ \alpha_S$

$$\begin{array}{ccc} S & \xrightarrow{f} & R \\ {\scriptstyle Q\circ\alpha_S}\downarrow & \swarrow {\scriptstyle g} & \\ T & & \end{array}$$

it will be shown that $f\colon S \to R$ is the holomorphic mapping in Theorem 1. It will then be shown that $g\colon R \to T$ is the Albanese mapping of R into its Albanese torus $A(R) = T$. Hence $Q \circ \alpha_s(S)$ is biholomorphic to R and f can be taken to be $Q \circ \alpha_S$.

Notationally, for $D = \sum_{j=1}^k n_j D_j$, let $F_j\colon N_j \to D_j$ be the normalization of D_j and let $i_j\colon D_j \to S$ be inclusion. Choose base points $p_j \in N_j$ to obtain the Albanese mappings of N_j, $\alpha_j\colon N_j \to A(N_j)$ for each j. Let $p = i_1 \circ F_1(p_1)$ be the base point in S for α_S. By the universal property of the Albanese mapping, there are Lie group homomorphisms $q_j\colon A(N_j) \to A(S)$ and translations $t_j \in A(S)$ with $(q_j \circ \alpha_j) + t_j = \alpha_S \circ (i_j \circ F_j)$. It is easily seen that $t_j \in \operatorname{span}_{j=1}^k \langle q_j(A(N_j))\rangle$.

Select a basis $\{\phi_1, \ldots, \phi_q\}$ for $H^0(S, \Omega^1_S)$ where $\phi_q = \phi$ is the 1-form pulling back to zero on D. Choosing a basis $\{\gamma_1, \ldots, \gamma_{2q}\}$ for the free part of $H_1(S, \mathbf{Z})$ yields a lattice Λ in $H^0(S, \Omega^1_S)^*$ generated by λ_j, $j = 1, \ldots, 2q$, where $\lambda_j(\phi_i) = \int_{\gamma_j} \phi_i$ for $i = 1, \ldots, q$. The assumption that S is Kähler implies that Λ is a closed lattice of rank $2q$ over $\mathbf{Z}$ in $H^0(S,\Omega^1_S)$. Then $A(S) = H^0(S,\Omega^1_S)^* \bmod \Lambda$ and $\alpha_S(x) = \sum_{i=1}^q (\int_p^x \phi_i)\phi_i^* \bmod \Lambda$ where $\{\phi_i^*\}_{i=1}^q$ is the dual basis to $\{\phi_i\}_{i=1}^q$.

Recall that $(i_j \circ F_j)^*\colon H^0(S,\Omega^1_S) \to H^0(N_j,\Omega^1_{N_j})$ induces

$$(i_j \circ F_j)^{*t}\colon H^0(N_j,\Omega^1_{N_j})^* \to H^0(S,\Omega^1_S)^*$$

and this in turn induces the Lie group homomorphism $q_j\colon A(N_j) \to A(S)$. Since $F_j^*(i_j^*\phi_q) = F_j^*(i_j^*\phi) = F_j^*(0) = 0$, $(i_j \circ F_j)^{*t}(H^0(N_j,\Omega^1_{N_j}))^* \subseteq \{z_q = 0\}$ where z_q is the coefficient of ϕ_q^* in $H^0(S,\Omega^1_S)^*$. Hence $q_j(A(N_j)) \subseteq \{z_q = 0\} \bmod \Lambda$ and, since S is Kähler $q_j(A(N_j))$ is a subtorus of $A(S)$ which is not all of $A(S)$. It is an easy consequence that $T_1 := \operatorname{span}_{j=1}^k \langle q_j(A(N_j))\rangle$ is a subtorus contained in $\{z_q = 0\}$ $\bmod \Lambda$. Defining $T = A(S)/T_1$ with $Q\colon A(S) \to T$ the quotient map, we have

$$\alpha_S(D_j) = \alpha_S(i_j \circ F_j(N_j)) = q_j(\alpha_j(N_j)) + t_j \subseteq q_j(A(N_j)) + t_j \subseteq T_1 + t_j \subseteq T_1$$

for all j. Hence $Q \circ \alpha_S(\bigcup_{j=1}^k D_j) \subseteq Q(T_1) = 0$. Since $\alpha_S(x) = \sum_{i=1}^q (\int_p^x \phi_i)\phi_i^*$ $\bmod \Lambda$ and since $\int_p^x \phi_q$ is not identically zero, $\alpha_S(S) \not\subset \{z_q = 0\} \bmod \Lambda$. So $\alpha_S(S) \not\subset$ T_1 and $Q \circ \alpha_S(S) \neq 0$. This fact and the fact that $Q \circ \alpha_S(S)$ is a connected analytic set imply $Q \circ \alpha_S(S)$ has dimension at least one. It remains to show that

PROPOSITION 1. *$Q \circ \alpha_S\colon S \to T$ has one-dimensional image.*

PROOF. Assume to the contrary that $Q \circ \alpha_S$ has two-dimensional image. We then show that $\bigcup_{j=1}^k D_j$ is contained in an exceptional set in S, contradicting the fact that $D \cdot D \geq 0$.

Recall $Q \circ \alpha_S(\bigcup_{j=1}^k D_j) = 0$. Let F be the connected component of $(Q \circ \alpha_S)^{-1}(0)$ containing $\bigcup_{j=1}^k D_j$. For

$$\begin{array}{ccc} S & \xrightarrow{f} & R \\ {\scriptstyle Q\circ\alpha_S}\downarrow & \swarrow {\scriptstyle g} & \\ T & & \end{array}$$

the Stein factorization of $Q \circ \alpha_S$, let $p' = f(F)$. We remark that, since S and T are normal, R is normal. Let $\mathscr{S}$ be an open neighborhood of p' in R such that $f^{-1}(\mathscr{S}) \cap [$ the one-dimensional fibers of $f] = F$ and $f^{-1}(\mathscr{S}) \cap [(Q \circ \alpha_S)^{-1}(0)] = F$. For $U := f^{-1}(\mathscr{S})$ we work with the localized Stein factorization

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathscr{S} \\ {\scriptstyle Q\circ\alpha_S}\downarrow & \swarrow {\scriptstyle g} & \\ T & & \end{array}$$

$\mathscr{S}$ is irreducible since it is normal. By choosing $\mathscr{S}$ sufficiently small, and by normality and two-dimensionality of $\mathscr{S}$, $p' = f(F)$ is the only possible singularity of $\mathscr{S}$.

Observe next that F is an exceptional set in U via the map f. This is done in two steps. First, F is nowhere discrete and satisfies $f\colon (U \backslash F) \to \mathscr{S} \backslash p'$ is a biholomorphism. The latter follows from Zariski's Main Theorem [**10**]. The second step is to observe that for all V open $\subset \mathscr{S}$ $f^*\colon \Gamma(V,\mathscr{O}) \to \Gamma(f^{-1}(V),\mathscr{O})$ is an isomorphism. f^* clearly injects and it is surjective by normality of $\mathscr{S}$. So F is exceptional in U.

F being an exceptional set is characterized by the fact that the matrix $(F_i \cdot F_j)$ of the intersection pairings of the irreducible components F_i of F is negative definite

[4]. By assuming $Q \circ \alpha_S$ had two-dimensional image, we have concluded $(F_i \cdot F_j)$ is negative definite. Recalling that $\bigcup_{j=1}^k D_j \subset F$ where $D = \sum_{j=1}^k n_j D_j$, let $F_j := D_j$ for $j = 1, \dots, k$. Then

$$0 \le D \cdot D = \sum_{i,j}^{k} n_i n_j D_i \cdot D_j$$
$$= (n_1, n_2, \dots, n_k, 0, \dots, 0)(F_i \cdot F_j)(n_1, n_2, \dots, n_k, 0, \dots, 0)^t.$$

This contradicts the fact that $(F_i \cdot F_j)$ is negative definite, unless $Q \circ \alpha_S(S)$ has dimension one, giving the proposition. □

To continue the proof of the theorem, observe that R is smooth, as it is normal and one-dimensional. It remains to show that the 1-form ϕ is a pullback via f of some holomorphic 1-form ϕ_R on R. The mappings $T_1 \xrightarrow{i} A(S) \xrightarrow{Q} T$ give corresponding mappings of covering spaces:

$$\begin{array}{ccccc} \mathbf{C}^n & \xrightarrow{\mathscr{I}} & \mathbf{C}^q & \xrightarrow{\mathscr{Q}} & \mathbf{C}^{q-n} \\ \downarrow & & \downarrow & & \downarrow \\ T_1 & \xrightarrow{i} & A(S) & \xrightarrow{Q} & T \end{array}$$

where $q = \dim A(S)$ and $n = \dim T_1$. Since $T_1 \subset \{z_q = 0\} \bmod \Lambda$, $\mathscr{I}(\mathbf{C}^n) \subseteq \{z_q = 0\} \subseteq \mathbf{C}^q$. So the qth coordinate z_q on $\mathbf{C}^q$ induces a coordinate z on $\mathbf{C}^{q-n}$ and thus on T. So $dz_q = Q^*(dz)$ in $H^0(A(S), \Omega^1_{A(S)})$. Now $\alpha_S^*(dz_q) = \phi_q = \phi$ by the choice of basis $\{\phi_1, \dots, \phi_q = \phi\}$ of $H^0(S, \Omega^1_S)$. Since $Q \circ \alpha_S = g \circ f$ in the Stein factorization, $\phi = \alpha_S^*(dz_q) = \alpha_S^*(Q^*(dz)) = f^*(g^*(dz))$. Letting $\phi_R = g^*(dz)$ gives the desired holomorphic form on R.

Before completing the proof of the theorem we show that T is the Albanese torus of R and

PROPOSITION 2. *$g\colon R \to T$ is the Albanese mapping of R. Hence $Q \circ \alpha_S(S)$ is smooth, biholomorphic to R and f can be chosen to be $Q \circ \alpha_S$.*

PROOF. One has the commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{f} & R & \xrightarrow{g} & T \\ \alpha_S \downarrow & & \downarrow \alpha_R & & \downarrow \mathrm{id} \\ A(S) & \xrightarrow{h} & A(R) & \xrightarrow{p} & T \end{array}$$

where h is induced by $\alpha_R \circ f$ and the universal property of $A(S)$, while p is induced by g and the universal property of $A(R)$. By appropriately selecting the base point of α_R, we can assume h and p are Lie group homomorphisms. Now h must be surjective, otherwise $h(A(S))$ is the Albanese torus of R. Since $p \circ h \circ \alpha_S = g \circ f = Q \circ \alpha_S$ one has $p \circ h = Q$ on $A(S)$. As Q is surjective, p is. Also $T_1 = \ker Q = \ker(p \circ h) \subset \ker h$ implies $\ker h = \ker(p \circ h) = T_1$. Thus $\ker p = 0$ and $p\colon A(R) \to T$ is a Lie group biholomorphism. Since the Albanese mapping of a Riemann surface to its Albanese torus is an embedding, $g(R) = Q \circ \alpha_S(S) = p \circ \alpha_R(R)$ is smooth and biholomorphic to R. □

To continue the proof of the theorem, we drop the assumption that

$$D = \sum_{j=1}^{k} n_j D_j$$

is connected. If D is not connected, find some connected component D_c of D with $D_c \cdot D_c \geq 0$ and apply the above results to obtain the map $f: S \to R$. Now each irreducible component of D is contained in a fiber of f (otherwise $f(D_j) = R$ for some j and $i_j^*\phi = i_j^* f^* \phi_R$ is nonzero on D_j where $i_j: D_j \to S$ is inclusion). In particular each connected component D_c of D is contained in a fiber of f. A theorem of Zariski [1] gives $D_c \cdot D_c \leq 0$. Since $D \cdot D \geq 0$ every connected component of D satisfies $D_c \cdot D_c = 0$ and so $D \cdot D = 0$. The result of Zariski then gives that, since each connected component D_c has self-intersection 0, each D_c is a rational multiple of the divisor associated to the fiber of f containing D_c. This finishes the theorem in the case S is Kähler.

2. Proof of Theorem 1: The non-Kähler case. To handle the non-Kähler case, note that by Kodaira's classification of compact surfaces and by the fact that $\phi \in H^0(S, \Omega_S^1) \neq 0$, S must be elliptic. Let $\pi: S \to R$ be the elliptic map onto the base curve R. There are two cases to handle:

(1) $\phi \in \pi^* H^0(R, \Omega_R^1)$,

(2) $\phi \in H^0(S, \Omega_S^1) \backslash \pi^* H^0(R, \Omega_R^1)$.

Case (1) is easily dealt with by letting $f = \pi$ and observing that all of the conclusions of the theorem are then satisfied by f. In case (2) it will be shown that if an elliptic surface S satisfies the hypotheses of the theorem along with the requirement that the 1-form $\phi \in H^0(S, \Omega_S^1) \backslash \pi^* H^0(R, \Omega_R^1)$, then S must be Kähler. It follows that case (2) is vacuous.

So assume S is elliptic, satisfies the hypotheses of the theorem, and

$$\phi \in H^0(S, \Omega_S^1) \backslash \pi^* H^0(R, \Omega_R^1).$$

We rely on the fact that, since ϕ is a nonpullback holomorphic 1-form on an elliptic surface, ϕ can never vanish on S, every fiber of π is a smooth torus, and away from multiple fibers S has the structure of a principal torus bundle with constant transition functions [**8**]. Since $d\phi = 0$, $\ker \phi$ defines an integrable distribution on S. Each irreducible component D_j of D is a leaf of the resulting foliation. Hence each D_j is smooth and intersects no other irreducible component of D. Also each irreducible component D_j meets every fiber of π transversely. This follows since, if D_j meets any fiber F of π tangentially at $p \in D_j \cap F$, then ϕ pulls back to zero at p on F and, since F is a torus, ϕ pulls back to zero identically on F. This implies that ϕ pulls back to zero on every fiber of π and therefore that $\phi \in \pi^* H^0(R, \Omega_R^1)$ [**8**], contrary to the assumption of case (2).

Select D_1 the first irreducible component of D. $\pi: D_1 \to R$ is unramified, except at multiple fibers of π, by transversality of D_1 to fibers of π. Let S have multiple fibers $B_1, \ldots, B_J$ where B_i has multiplicity m_i. Define $S' = S \backslash \bigcup_{i=1}^J B_i$, $D_1' = D_1 \cap S'$, and $X' = \{(d,p) \in D_1' \times S' | \pi(d) = \pi(p)\}$. As mentioned earlier, S' is a principal torus bundle over $\pi(D_1')$ having constant transition functions and fiber $F = T$ a torus. This implies X' is a principal torus bundle over D_1', via first factor projection π_1, with constant transition functions and fiber T. Second coordinate projection $\pi_2: X' \to S'$ gives X' as an m-fold cover of S' where $m := F \cdot D_1$, for F a general fiber of π. X' is seen to be biholomorphic to $D_1' \times T$ by observing that, if $i_1: D_1 \to S$ is inclusion and if $\mathscr{I}: D_1' \to X'$ is the inclusion given by $\mathscr{I}(d) = (d, i_1(d))$, then $\mathscr{I}(D_1')$ gives a zero section of X'.

We next show that the m-fold cover $X' \simeq D_1' \times T$ of S' can be completed in a natural way to an m-fold cover X of S, with X biholomorphic to $D_1 \times T$. This then shows S is algebraic, since its cover $D_1 \times T$ is. Hence S is Kähler and case (2) is vacuous as claimed, proving Theorem 1 in general.

X is constructed by taking suitable covers Y_i of neighborhoods of the multiple fibers B_i, and patching the Y_i into X'. Examine S' near a multiple fiber B_i. Let $S_i^* = \pi^{-1}(\Delta_i)\backslash B_i \subset S'$ where Δ_i is a coordinate disc neighborhood in R centered at $\pi(B_i)$. Let u_i be a fixed coordinate function on Δ_i with $u_i(\pi(B_i)) = 0$. Define $X_i^* = \pi_2^{-1}(S_i^*) \subset X'$; explicitly $X_i^* = \{(d,p) \in (S_i^* \cap D_1) \times S_i^* \mid \pi(d) = \pi(p)\}$. Let D_{in} be the components of $\pi^{-1}(\Delta_i) \cap D_1$ where n ranges from 1 to m/m_i. (Recall that $m := F \cdot D_1 = (m_i B_i) \cdot D_1$ implies $m_i \mid m$ where m_i is the multiplicity of B_i.) Let $D_{in}^* = (S_i^* \cap D_{in})$. Also let the components of X_i^* be $X_{in}^* = \{(d,p) \in D_{in}^* \times S_i^* \mid \pi(d) = \pi(p)\}$ and let

$$Y_i = \{(p,z) \in \pi^{-1}(\Delta_i) \times \mathscr{O}_{\pi^{-1}(\Delta_i)} \mid z \in \mathscr{O}_p,\ z^{m_i} = u_i \circ \pi\}$$

where $\mathscr{O}_{\pi^{-1}(\Delta_i)}$ is the sheaf of germs of holomorphic functions on $\pi^{-1}(\Delta_i)$. Y_i has the structure of an m_i-fold cover of $\pi^{-1}(\Delta_i)$ under first factor projection $\rho_i\colon Y_i \to \pi^{-1}(\Delta_i)$, while each X_{in}^* has the structure of an m_i-fold cover of $S_i^* \subset \pi^{-1}(\Delta_i)$ under second factor projection π_2. These covering structures on Y_i and X_{in}^* in fact coincide over S_i^*.

To see this, let u_{in} be a fixed coordinate function on D_{in} satisfying $(u_{in})^{m_i} = u_i$. Define $\gamma_{in}\colon X_{in}^* \to Y_i$ by $\gamma_{in}(d,p) = (p, z_d)$ where z_d is the function element around p given by (i) $z_d^{m_i} = u_i \circ \pi$ and (ii) $z_d(p) = u_{in}(d)$. γ_{in} gives a biholomorphism of X_{in}^* with $\rho_i^{-1}(S_i^*) \subset Y_i$, as $\gamma_{in}^{-1}(p,z) = (u_{in}^{-1}(z(p)), p)$. Since $\pi_2 \circ \gamma_{in}^{-1} = \rho_i$, the covering structures on X_{in}^* and Y_i agree over S_i^*.

$$\begin{array}{ccc} X_{in}^* & \xrightarrow{\gamma_{in}} & Y_i \\ \pi_2 \downarrow & & \downarrow \rho_i \\ S_i^* & \xrightarrow{\text{id}} & S_i \end{array}$$

Hence for each $n = 1, \ldots, m/m_i$ a copy of Y_i can be patched into $X_{in}^* \subset X'$ via the map γ_{in}^{-1}. Do this for each multiple fiber B_i to obtain the resulting covering space X of S, with $\rho\colon X \to S$ the covering map induced by π_2 on X' and ρ_i on Y_i.

It must be shown next that X is biholomorphic to $D_1 \times T$. To accomplish this, it is first shown that the inclusion $\mathscr{I}$ of D_1' into X' extends to an inclusion I of D_1 into X. This is done locally by observing that the inclusion of D_{in}^* into X_{in}^* given by $\mathscr{I}(d) = (d, i_1(d))$ extends to an inclusion of D_{in} into Y_i under γ_{in}. Let u_{in} be the fixed coordinate on D_{in} as above. Extend u_{in} to w_{in} on an open neighborhood of $i_1(D_{in})$ in $\pi^{-1}(\Delta_i)$ by requiring w_{in} to be constant on fibers of π and that $w_{in} \circ i_1 = u_{in}$ on D_{in}. Define $I_{in}\colon D_{in} \to Y_i$ by $I_{in}(d) = (i_1(d), (w_{in})_d)$ where $(w_{in})_d$ is the germ of w_{in} at $i_1(d)$. Note that for $I_{in}(d) \in \rho_i^{-1}(S_i^*) \subset Y_i$,

$$\gamma_{in}^{-1}(I_{in}(d)) = \gamma_{in}^{-1}(i_1(d), (w_{in})_d) = (u_{in}^{-1}(w_{in}(i_1(d))), i_1(d)) = (d, i_1(d)) = \mathscr{I}(d).$$

So $I_{in} = \gamma_{in} \circ \mathscr{I}$ on D_{in}^* for each i, n. The I_{in} together with $\mathscr{I}$ define the inclusion $I\colon D_1 \to X$. $I(D_1)$ will be the zero section in the trivialization $X \simeq D_1 \times T$.

The map $\mu_{in}\colon Y_i \to D_{in}$ given by $\mu_{in}(p,z) = u_{in}^{-1}(z(p))$ exhibits Y_i as a deformation of tori over D_{in} (there is no multiple fiber over $D_{in} \cap B_i$). Since μ_{in}^{-1} of any point in D_{in}^* is T, $\mu_{in}^{-1}(D_{in} \cap B_i) = T$. Hence μ_{in} is trivial over D_{in}. Note that μ_{in}

satisfies $\pi_1 = \mu_{in} \circ \gamma_{in} \colon X^*_{in} \to D^*_{in}$ where π_1 is first factor projection on $X^*_{in} \subseteq X'$. This follows since

$$\mu_{in} \circ \gamma_{in}(d,p) = \mu_{in}(p, z_d) = u_{in}^{-1}(z_d(p)) = u_{in}^{-1}(u_{in}(d)) = d = \pi_1(d,p).$$

So $\pi_1 \colon X' \to D'_1$ extends to a global projection $\mu \colon X \to D_1$ with fiber T and no multiple fibers. Recall that $\rho \colon X \to S$ was the covering map extending $\pi_2 \colon X' \to S'$. Now $\rho^*\phi$ is a holomorphic 1-form on X which is not an element of $\mu^* H^0(D_1, \Omega^1_{D_1})$, as $\rho^*\phi$ pulls back to zero on $I(D_1) \subset X$. Hence X is a principal torus bundle over D_1 having constant transition functions and fiber T [**8**]. Letting $I(D_1)$ be the zero section of X over D_1 gives X is biholomorphic to $D_1 \times T$ and finishes the proof of the theorem.

3. The higher dimensional projective case. In this section we generalize Theorem 1 to the case in which M is a compact projective algebraic manifold of arbitrary dimension, where M carries a holomorphic 1-form pulling back to zero on a divisor D which satisfies a certain nonnegativity condition.

THEOREM 2. *Let M be a compact projective manifold of dimension n, with ω the Kähler form induced by $\mathbf{P}^N$. If M carries a holomorphic 1-form ϕ which pulls back to zero on a divisor $D = \sum n_j D_j$ with $\int_M c_1^2(D) \wedge \omega^{n-2} \geq 0$ then there exists a holomorphic map $f \colon M \to R$ onto a Riemann surface of nonzero genus with*

(1) *$\phi = f^*(\phi_R)$ for some $\phi_R \in H^0(R, \Omega^1_R)$.*
(2) *f has connected fibers.*
(3) *Each component of D is setwise contained in a fiber of f.*
(4) *$\int_M c_1^2(D) \wedge \omega^{n-2} = 0$.*

PROOF. As in Theorem 1, the inclusions of the D_j into M give rise to mappings $i_j \colon N_j \to M$ from the desingularizations N_j into M. These induce mappings $q_j \colon A(N_j) \to A(M)$ on the Albanese tori. For $T_1 = \operatorname{span}_{j=1}^k \langle q_j(A(N_j))\rangle$ the quotient map $Q \colon A(M) \to A(M)/T_1$ composed with the Albanese mapping $\alpha_M \colon M \to A(M)$ gives the desired mapping $f = Q \circ \alpha_M \colon M \to A(M)/T_1$.

The main issue is to show that $f(M)$ is one dimensional, which by using Stein factorization, normality, and Proposition 2, will give that $f(M)$ is smooth. Since $f(M)$ cannot be zero dimensional, we only need show that $\dim f(M) \geq 2$ is prohibited. This is done by slicing M with hyperplane sections.

If $\dim f(M)$ were 2 or more, one could choose a hyperplane H_1 such that

(a) $M \cap H_1$ is smooth.
(b) H_1 is transverse to D at some point.
(c) $f(M \cap H_1)$ is of maximal dimension.

One could continue slicing with hyperplanes where the requirements in choosing the jth hyperplane H_j are

(a) $M \cap H_1 \cap H_2 \cap \cdots \cap H_j$ is smooth.
(b) H_j is transverse to $D \cap H_1 \cap \cdots \cap H_{j-1}$ at some point.
(c) $f(M \cap H_1 \cap \cdots \cap H_j)$ is of maximal dimension.

Then $M \cap H_1 \cap H_2 \cap \cdots \cap H_{n-2} := S$ is a surface and $f(M \cap H_1 \cap \cdots \cap H_{n-2})$ has dimension 2. So $D \cap H_1 \cap \cdots \cap H_{n-2}$ is contained in an exceptional set in S and hence the divisor $\widetilde{D}$ on S induced by D satisfies

$$\widetilde{D}\cdot\widetilde{D}=\int_{M\cap H_1\cap\cdots\cap H_{n-2}} c_1^2(D) = \int_M c_1^2(D)\wedge c_1^{n-2}(H) = \int_M c_1^2(D)\wedge\omega^{n-2}<0.$$

This contradicts the hypothesis that $\int_M c_1^2(D)\wedge\omega^{n-2}\geq 0$ giving that $\dim f(M) = 1$. The remainder of the argument parallels that of Theorem 1. □

4. Remarks and applications. Theorems 1 and 2, in many cases, can be interpreted as theorems about zero sets of holomorphic 1-forms. The theorems give, for D with appropriate nonnegativity condition and a holomorphic 1-form pulling back to zero on D, a holomorphic mapping $f\colon M\to R$ and a holomorphic 1-form ϕ_R on R with $\phi = f^*(\phi_R)$. If the genus $g(R)\geq 2$ then ϕ_R vanishes at some point $p\in R$. Letting D_0 be the divisor associated to $f^{-1}(p)$, one has $D_0\cdot D_0 = 0$ and ϕ vanishes on D_0. Thus one often expects a divisor D_0 of nonnegative self-intersection on which ϕ actually vanishes (as opposed to pulling back to zero). Even if $g(R) = 1$ multiple fibers will give such a D_0. However, for $g(R) = 1$ such a D_0 does not exist unless there is a fiber all of whose irreducible components are multiple.

Theorems 4 and 5 will show that the genus of an irreducible component of the zero set of a holomorphic 1-form on a compact complex surface S is bounded above by Chern numbers of S. We outline this result later. First, defining the genus of a curve $C=\sum_{j=1}^k n_jC_j$ to be $g(C) := \sum_{j=1}^k g(NC_j)$ where $g(NC_j)$ is the genus of the normalization of C_j, we show the following partial converse:

THEOREM 3. *Let S be a compact complex surface with the first Betti number $b_1(S)$. If $C\subset S$ is any connected curve of nonnegative self-intersection satisfying $g(C)\leq\frac{1}{2}b_1(S)-2$ then $C\cdot C = 0$ and C is setwise contained in the zero set of some holomorphic 1-form on S.*

PROOF. Since $g(C)\leq\frac{1}{2}b_1(S)-2$ there is at least one holomorphic 1-form on S which pulls back to zero on C. To see this, if $b_1(S)$ is even then $b_1(S) = 2h^{1,0}(S)$ where $h^{1,0}(S) = \dim H^0(S,\Omega_S^1)$, otherwise $b_1(S) = 2h^{1,0}(S)+1$ [**5**]. In either case $g(C)=\sum_{j=1}^k g(NC_j)$ being integral and less than or equal to $\frac{1}{2}b_1(S)-2$ implies $\sum_{j=1}^k g(NC_j)\leq h^{1,0}(S)-2$. This gives that

$$\dim\bigcap_{j=1}^k \ker i_j^* = \dim\bigcap_{j=1}^k \ker F_j^* i_j^* \geq h^{1,0}(S)-\sum_{j=1}^k g(NC_j)\geq 2$$

where $i_j\colon D_j\to S$ is inclusion, $F_j\colon NC_j\to C_j$ is normalization, and $\ker i_j^* = \{\phi\in H^0(S,\Omega_S^1)\mid i_j^*\phi = 0\} = \ker F_j^*i_j^*$. So $\bigcap_{j=1}^k\ker i_j^*$ nonzero gives a holomorphic 1-form pulling back to zero on C.

Apply Theorem 1 to obtain a holomorphic map $f\colon S \to R$ with $C \cdot C = 0$ and C setwise contained in a fiber of f, say $f^{-1}(p)$. The object is to show $g(R) \geq 2$. Then there is a holomorphic 1-form ψ on R vanishing at p, giving that $f^*(\psi)$ vanishes on $f^{-1}(p)$ and therefore on C.

In the case S is Kähler, we have, in the notation of §1,

$$\begin{aligned} g(R) &= \dim A(R) = \dim A(S) - \dim \operatorname{span}_{j=1}^{k}\langle q_j(A(NC_j))\rangle \\ &\geq \dim H^0(S, \Omega_S^1) - \sum_{j=1}^{k} \dim q_j(A(NC_j)) \\ &\geq \dim H^0(S, \Omega_S^1) - \sum_{j=1}^{k} \dim A(NC_j) \\ &= \frac{1}{2} b_1(S) - \sum_{j=1}^{k} g(NC_j) \geq 2. \end{aligned}$$

In the case S is non-Kähler, and therefore is elliptic, we have that $h^{1,0}(S) = g(R) + \delta$ where $\delta = 0$ or 1. This yields

$$\sum_{j=1}^{k} g(NC_j) \leq h^{1,0}(S) - 2 \leq g(R) - 1. \tag{4.1}$$

To show $g(R) \geq 2$, (4.1) allows reduction to the case $\sum_{j=1}^{k} g(NC_j) = 0$. Since $C \cdot C = 0$, $\bigcup_{j=1}^{k} C_j$ is not an exceptional set in S, and so $\bigcup_{j=1}^{k} C_j$ is a singular fiber consisting of rational curves. Any elliptic surface containing a singular fiber consisting of rational curves has every holomorphic 1-form as an element of $f^* H^0(R, \Omega_R^1)$ [**8**]. Hence $h^{1,0}(S) = g(R)$ and (4.1) becomes $0 \leq g(R) - 2$. □

For completeness, we include the statement of two theorems bounding the genus of an irreducible component of a one-dimensional zero set of a holomorphic 1-form on a compact surface. Full details of the proof appear in [**9**], but the main idea is sketched here.

THEOREM 4. *Let S be a compact complex surface and let C be an irreducible curve contained in the zero set of a holomorphic 1-form ϕ on S. Then the genus of C is bounded by $g(C) \leq 1 + \frac{3}{2}\max(0, c_2(S))$.*

Sharper bounds on $g(C)$ which depend on the cateogry of the surface S are given by

THEOREM 5. *Let S be a compact complex surface free from exceptional curves, and let C be an irreducible curve contained in the zero set of a holomorphic 1-form ϕ on S. Let $H_\phi = \{\psi \in H^0(S, \Omega_S^1) \mid \psi \wedge \phi \equiv 0\}$, $h_\phi = \dim H_\phi$, and $h^{1,0}(S) = \dim H^0(S, \Omega_S^1)$.*

Then $g(C)$, the genus of C, is bounded by one of the following:

(1)	$g(C) = 0$	*if S is ruled,*
(2)	$g(C) \leq 1$	*if S is elliptic,*
(3)	$g(C) \leq 1 + \frac{1}{2}c_1^2(S)$	*if $h^{1,0}(S) \geq 2$ and there exists $\psi \in H^0(S, \Omega_S^1)$ with $\phi \wedge \psi \not\equiv 0$,*
(4)	$g(C) \leq 1 + \frac{c_2(S)}{4(h_\phi - 1)} \leq 1 + \frac{1}{4}c_2(S)$	*if $h^{1,0}(S) \geq 2$, S is nonruled, and $h_\phi \geq 2$ (there exists $\psi \in H^0(S, \Omega_S^1)$ with $\phi \wedge \psi \equiv 0$)*
(5)	$g(C) \leq 1 + \frac{1}{2}c_2(S)$	*if $h^{1,0}(S) = 1$ and S is neither elliptic nor ruled.*

Hence, $g(C) \leq 1 + \frac{1}{2}\max(0, c_1^2(S), c_2(S))$.

The proof of these theorems is based on the fact that $g(C) \leq \pi(C)$, where $\pi(C) = 1 + \frac{1}{2}(K \cdot C + C \cdot C)$ is the virtual genus of C and K is the canonical divisor of S. Theorem 1 gives that $C \cdot C \leq 0$ for C contained in the zero set of a holomorphic 1-form. This gives that $g(C) \leq 1 + \frac{1}{2}K \cdot C$. One then bounds $K \cdot C$ or $K \cdot C + C \cdot C$ by containing C in either the canonical divisor K or the fiber of a holomorphic map of S onto a Riemann surface. If C is contained in an effective canonical divisor and S is free from exceptional curves, then $0 \leq K \cdot C \leq K \cdot K$ [**5**], giving $g(C) \leq 1 + \frac{1}{2}K \cdot K$. If C is contained in the fiber of an appropriate holomorphic map, one relies on the fact that $g(C) \leq g(F)$ where F is a regular fiber. The remaining cases to handle are: S is ruled giving $g(C) = 0$; S is elliptic giving $g(C) \leq 1$; S maps holomorphically to a Riemann surface R with $g(R) \geq 2$ giving $c_2(S) \geq 4(g(F) - 1)(g(R) - 1)$ and hence $1 + c_2(S)/4 \geq g(F) \geq g(C)$; and finally S maps to a one-dimensional Albanese torus with C in a fiber and $K \cdot C \leq c_2(S)$. As the above five cases are comprehensive, one easily constructs a universal bound $g(C) \leq 1 + \frac{1}{2}\max(0, c_1^2(S), c_2(S))$ for S minimal and deduces $g(C) \leq 1 + \frac{3}{2}\max(0, c_2(S))$ for general S [**9**].

Finally, in the way of examples of the fiberings of Theorem 1, we remark that in [**7**] it has been shown that among minimal surfaces of general type there is a class of surfaces admitting holomorphic maps to higher genus Riemann surfaces for which the Chern ratios $c_1^2(S)/c_2(S)$ assume all rational values in the interval $[\frac{1}{5}, 3]$.

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